

SEPARABLY INJECTIVE C_σ -SPACES

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ABSTRACT. We show that a (complex) C_σ -space is separably injective if and only if it is linearly isometric to the Banach space $C_0(\Omega)$ of complex continuous functions vanishing at infinity on a substonean locally compact Hausdorff space Ω .

1. INTRODUCTION

Recently, separably injective Banach spaces have been studied in depth by Avilés, Cabello Sánchez, Castillo, González and Moreno in [2, 3, 4], where one can find a number of interesting examples of these spaces despite the scarcity of examples of injective Banach spaces. In contrast to the fact that 1-injective Banach spaces are isometric to the Banach space $C(\Omega)$ of continuous functions on a compact Hausdorff space Ω [12, 14], 1-separably injective Banach spaces need not be complemented in $C(\Omega)$. In view of this, a natural question arises: when is a 1-complemented subspace of $C(\Omega)$ separably injective? We address this question in this paper.

A Banach space V is 1-*separably injective* if every continuous linear map $T : Y \rightarrow V$ on a closed subspace Y of a *separable* Banach space Z admits a norm preserving extension to Z . It is known that $C(\Omega)$ itself is separably injective if and only if Ω is an F -space [2]. Abelian C^* -algebras with identity are of the form $C(\Omega)$ for some Ω . The ones without identity can be represented as the algebra $C_0(S)$ of complex continuous functions vanishing at infinity on a locally compact Hausdorff space S and it has been shown lately that $C_0(S)$ is separably injective if and only if S is substonean [7]. Following [11], we call S *substonean* if any two disjoint open σ -compact subsets of S have disjoint compact closures. The compact substonean spaces are exactly the F -spaces defined in [8, 18]. However, infinite discrete spaces are F -spaces without being substonean. We refer to [13, Example 5] for an example of a substonean space which is not an F -space.

Noting that the class of 1-complemented subspaces of $C(\Omega)$ is identical to that of C_σ -spaces [15, Theorem 3] (see also Remark 2.1), our question amounts to asking for a characterisation of separably injective C_σ -spaces. We give a complete answer by showing that a C_σ -space is separably injective if and only if it is linearly isometric to the function space $C_0(S)$ on a substonean locally compact Hausdorff space S .

In what follows, all Banach spaces are over the complex field and we will denote by $C_0(K)$ the C^* -algebra of complex continuous functions vanishing at infinity on a locally compact Hausdorff space K . If K is compact, we omit the subscript 0. Given a function $g \in C_0(K)$, we denote by \bar{g} the complex conjugate of g .

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Let $\mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$ be the circle group. By a \mathbb{T} -space, we mean a locally compact Hausdorff space K equipped with a continuous group action

$$\sigma : (\alpha, \omega) \in \mathbb{T} \times K \mapsto \alpha \cdot \omega \in K.$$

A complex Banach space is called a *complex C_σ -space* if it is linearly isometric to a function space of the form $C_\sigma(K)$ for some \mathbb{T} -space K , defined by

$$C_\sigma(K) = \{f \in C_0(K) : f(\sigma(\alpha, \omega)) = \alpha f(\omega), \forall \omega \in K\}.$$

We note that the definition of a complex C_σ -space in [16] requires K to be compact.

To achieve our result, we make substantial use of the Jordan algebraic structure of $C_\sigma(K)$. Indeed, although $C_\sigma(K)$ lacks a C^* -algebraic structure, it is equipped with a triple product

$$\{f, g, h\} = f\bar{g}h \quad (f, g, h \in C_\sigma(K))$$

which turns it into a *JB*-triple* with many useful Jordan properties.

First, let us give a brief introduction to JB*-triples which generalise C^* -algebras. For further references and the geometric origin of JB*-triples, we refer to [6, 17, 19]. A complex Banach space V is a *JB*-triple* if it admits a continuous triple product

$$\{\cdot, \cdot, \cdot\} : V^3 \longrightarrow V$$

which is symmetric and linear in the outer variables, but conjugate linear in the middle variable, and satisfies

- (i) $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\} + \{a, b, \{x, y, c\}\};$
- (ii) the operator $a \square a : x \in V \mapsto \{a, a, x\} \in V$ has real numerical range and non-negative spectrum;
- (iv) $\|a \square a\| = \|a\|^2$

for $a, b, c, x, y \in V$. We always have $\|\{a, b, c\}\| \leq \|a\|\|b\|\|c\|$ and (i) is called the *Jordan triple identity*. A C^* -algebra A is a JB*-triple with the triple product

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a) \quad (a, b, c \in A).$$

More generally, the range of a contractive projection on a C^* -algebra is a JB*-triple (cf. [6, Theorem 3.3.1]), but not always a C^* -algebra. An element e in a JB*-triple is called a *tripotent* if $e = \{e, e, e\}$. Tripotents in C^* -algebras are exactly the partial isometries.

A subspace W of a JB*-triple V is called a *subtriple* if $a, b, c \in W$ implies $\{a, b, c\} \in W$. Closed subtriples of a JB*-triple are JB*-triples in the inherited norm and triple product. A *triple ideal* of V is a subspace $J \subset V$ such that $\{a, b, c\} \in J$ whenever one of a, b and c belongs to J . Given a closed triple ideal $J \subset V$, the quotient space V/J is a JB*-triple in the triple product

$$\{a + J, b + J, c + J\} := \{a, b, c\} + J.$$

Two elements $a, b \in V$ are said to be *orthogonal* to each other if $a \square b = 0$, where $a \square b$ is the continuous linear map $x \in V \mapsto \{a, b, x\} \in V$. Two subspaces $I, J \subset V$ are *orthogonal* if $I \square J := \{a \square b : a \in I, b \in J\} = \{0\}$.

The bidual V^{**} of a JB*-triple V carries a natural structure of a JB*-triple, with a unique predual, in which the triple product is separately weak* continuous and the natural embedding of V into V^{**} identifies V as a subtriple of V^{**} . Given a closed triple ideal $I \subset V$, the bidual I^{**} embeds as a weak* closed triple ideal in

V^{**} , which can be decomposed into an ℓ^∞ -sum $V^{**} = I^{**} \oplus_\infty J$ for some weak* closed triple ideal $J \subset V^{**}$, orthogonal to I^{**} [6, lemma 3.3.16].

A linear map $\varphi : V \rightarrow W$ between two JB*-triples is called a *triple homomorphism* if $\{\varphi(a), \varphi(b), \varphi(c)\} = \varphi\{a, b, c\}$ for $a, b, c \in V$. The triple isomorphisms between V and W are exactly the surjective linear isometries (cf. [6, Theorem 3.1.7, Theorem 3.1.20]).

A JB*-triple V is called *abelian* if its triple product satisfies

$$\{\{x, y, z\}, u, v\} = \{x, \{y, z, u\}, v\} = \{x, y, \{z, u, v\}\}$$

for all $x, y, z, u, v \in V$. An abelian C*-algebra is an abelian JB*-triple and so is $C_\sigma(K)$ in the triple product defined above. In fact, $C_\sigma(K)$ is a closed subtriple of $C_0(K)$.

By [5, Lemma 2.2, Theorem 3.7], an abelian closed subtriple V of a C*-algebra admits a composition series $(J_\lambda)_{0 \leq \lambda \leq \mu}$ of closed triple ideals, indexed by ordinals λ , such that the quotient $J_{\lambda+1}/J_\lambda$ is linearly isometric to an abelian C*-algebra, for $\lambda < \mu$. We recall that $(J_\lambda)_{0 \leq \lambda \leq \mu}$ is called a *composition series* if $J_0 = \{0\}$, $J_\mu = V$ and for a limit ordinal $\lambda \leq \mu$, the ideal J_λ is the closure of $\bigcup_{\lambda' < \lambda} J_{\lambda'}$.

2. JORDAN STRUCTURE IN C_σ -SPACES

We will make use of the abelian JB*-triple structure of C_σ -spaces to derive our result. To pave the way, we first present some detailed analysis of this structure.

Let V be an abelian closed subtriple of a C*-algebra (e.g. a C_σ -space) in this section. One can consider it as a subtriple of its bidual V^{**} via the natural embedding $v \in V \mapsto \widehat{v} \in V^{**}$, where $\widehat{v}(\psi) = \psi(v)$ for $\psi \in V^*$. By [5, 9], V^{**} is (isometric to and identified as) an *abelian* von Neumann algebra with identity denoted by $\mathbf{1}$ and involution $z \in V^{**} \mapsto z^* \in V^{**}$. The triple product in V^{**} is given by $\{a, b, c\} = ab^*c$. Each $\psi \in V^*$ can be viewed naturally as a functional of V^{**} . If ψ is a positive functional of V^{**} , then $\psi(z^*) = \overline{\psi(z)}$ for each $z \in V^{**}$. A positive functional $\psi \in V^*$ is called a *normal state* of V^{**} if $\psi(\mathbf{1}) = 1$. It is called *pure* if it is an extreme point of the norm closed convex set of normal states in V^* .

Let S be the set of all pure normal states of V^{**} , which are exactly the multiplicative normal states of V^{**} . Given a projection $p \in V^{**}$ and $s \in S$, we have $s(p) = 0$ or 1 since $s(p) = s(p^2) = s(p)^2$. If $u \in V^{**}$ is unitary, then $1 = s(\mathbf{1}) = s(u^*u) = |s(u)|^2$ for all $s \in S$. We equip S with the weak* topology of V^* and call it the *pure normal state space* of V^{**} .

The nonzero triple homomorphisms from V to \mathbb{C} are exactly the set $K = \text{ext } V_1^*$ of extreme points of the dual unit ball V_1^* , where $K \cup \{0\}$ is weak*-compact [9, Proposition 2.3, Corollary 2.4] and $S \subset K$.

For each $\omega \in K$ and tripotent $c \in V^{**}$, we have

$$\omega(c) = \omega(cc^*c) = \omega(c)|\omega(c)|^2$$

which implies $\omega(c) = 0$ or $\omega(c) \in \mathbb{T}$. We note that K is a \mathbb{T} -space with the natural \mathbb{T} -action

$$\sigma : (\alpha, \omega) \in \mathbb{T} \times K \mapsto \alpha\omega \in K$$

and we have $K = \{\alpha s : \alpha \in \mathbb{T}, s \in S\}$. In fact, each $\omega \in K$ has a *unique* representation $\omega = \alpha s$ for some $\alpha \in \mathbb{T}$ and $s \in S$, where $s = \overline{\omega(\mathbf{1})}\omega$. By [9, Theorem 1], the map

$$(2.1) \quad v \in V \mapsto \widehat{v}|_K \in C_\sigma(K)$$

is a surjective linear isometry, which enables us to identify V with the C_σ -space $C_\sigma(K)$.

Remark 2.1. Let $\pi : C(\Omega) \rightarrow C(\Omega)$ be a contractive projection. Then its image $\pi(C(\Omega))$ is an abelian closed subtriple in some C^* -algebra [10] and hence the previous discussion implies that it is a C_σ -space.

For each $a \in V \setminus \{0\}$, let $V(a)$ be the JB^* -subtriple generated by a in V . Then there is a surjective linear isometry and triple isomorphism

$$(2.2) \quad \phi : C_0(S_a) \rightarrow V(a) \subset V$$

which identifies $V(a)$ with the abelian JB^* -triple $C_0(S_a)$ of continuous functions vanishing at infinity on the triple spectrum $S_a \subset (0, \|a\|]$, where $S_a \cup \{0\}$ is compact [6, Theorem 3.1.12].

Let $\phi^{**} : C_0(S_a)^{**} \rightarrow V(a)^{**}$ be the bidual map and let i be the identity in the von Neumann algebra $C_0(S_a)^{**}$. Then $e = \phi^{**}(i)$ is the identity in the von Neumann algebra $V(a)^{**}$ with product and involution given by

$$x \cdot y = \{x, e, y\}, \quad x \in V(a)^{**} \mapsto \{e, x, e\} \in V(a)^{**}.$$

While this abelian von Neumann algebraic structure of $V(a)^{**}$ will be assumed throughout, it should be noted that $V(a)^{**}$ need not be a subalgebra of V^{**} in its natural embedding. Nevertheless, $V(a)^{**}$ can always be considered as a subtriple of V^{**} and the identity $e \in V(a)^{**}$ is a tripotent in V^{**} satisfying

$$(2.3) \quad \{e, a, e\} = \{\phi^{**}(i), \phi^{**}(\iota_a), \phi^{**}(i)\} = \phi^{**}\{i, \iota_a, i\} = a.$$

For each $\rho \in K$, viewed as a complex-valued triple homomorphism on $V(a)^{**}$, we have $\rho(e) = 0$ if and only if $\rho(a) = \rho\{e, e, a\} = 0$.

The norm closed triple ideal J_a generated by a in V contains $V(a)$ and is the norm closure of $\{a, V, a\}$. It has been shown in [5, Lemma 2.2] that J_a is linearly isometric to the abelian C^* -algebra $C_0(X_a)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space X_a . We need some detail here for later application. In fact, J_a is an abelian C^* -algebra with the same product and involution of $V(a)^{**}$, and $e \in V(a)^{**} \subset J_a^{**}$ is the identity of J_a^{**} , which can be seen from the following computation using (2.3).

$$\begin{aligned} \{\{a, x, a\}, e, \{a, y, a\}\} &= \{a, \{x, \{a, e, a\}, y\}, a\} \in \{a, V, a\}; \\ \{e, \{a, x, a\}, e\} &= ea^*xa^*e = ee^*ae^*xe^*ae^*e = ae^*xe^*a = \{x, a, a\} \in J_a. \end{aligned}$$

Given $\rho \in K = \text{ext } V_1^*$ with $\rho(e) = 1$, its restriction $\rho|_{J_a} \in J_a^*$ is a pure normal state of J_a^{**} . Conversely, each pure normal state φ of J_a^{**} is an extreme point of the closed unit ball of J_a^* and can be extended to an extreme point $\tilde{\varphi} \in \text{ext } V_1^*$ satisfying $\tilde{\varphi}(e) = 1$. Let

$$X_a = \{\rho|_{J_a} : \rho \in K, \rho(e) = 1\}$$

denote the pure normal state space of J_a^{**} which is locally compact in the weak* topology $w(J_a^*, J_a)$ of J_a^* .

We note that for each $\rho \in K$, we have $\rho(a) = 0$ if and only if $\rho(V(a)) = \{0\}$, which in turn is equivalent to $\rho(J_a) = \overline{\rho(\{a, V, a\})} = \{0\}$.

Lemma 2.2. *In the above notation, the set $K(e) = \{\rho \in K : \rho(e) = 1\}$ with the relative weak* topology of V^* is homeomorphic to $X_a = \{\rho|_{J_a} : \rho \in K, \rho(e) = 1\}$ in the topology $w(J_a^*, J_a)$. In particular, $K(e)$ is weak* locally compact in K .*

Proof. We show that the restriction map $\rho \in K(e) \mapsto \rho|_{J_a} \in X_a$ is a homeomorphism in these topologies. It is clearly continuous and surjective. Given $\rho, \rho' \in K$ such that $\rho|_{J_a} = \rho'|_{J_a}$, then $\rho(a) = \rho'(a) \neq 0$ since $\rho(e) = \rho'(e) = 1$. For any $v \in V$, we have $\{a, a, v\} \in J_a$ and hence $|\rho(a)|^2 \rho(v) = \rho\{a, a, v\} = \rho'\{a, a, v\} = |\rho'(a)|^2 \rho'(v)$, giving $\rho(v) = \rho'(v)$. This proves injectivity of the map.

Finally, to show that the inverse of the map is continuous, let $(\rho_\gamma|_{J_a})$ be a net converging to $\rho|_{J_a} \in X_a$. Then again, for each $v \in V$, we have $\rho_\gamma(a) \rightarrow \rho(a) \neq 0$ and $|\rho_\gamma(a)|^2 \rho_\gamma(v) = \rho_\gamma\{a, a, v\} \rightarrow \rho\{a, a, v\} = |\rho(a)|^2 \rho(v)$, which implies $\rho_\gamma(v) \rightarrow \rho(v)$, proving continuity. \square

Remark 2.3. The above lemma enables us to identify the pure normal state space X_a with $K(e)$ and write $X_a = \{\rho \in K : \rho(e) = 1\}$.

We retain the above notation in the sequel.

3. SEPARABLY INJECTIVE C_σ -SPACES

We characterize separably injective C_σ -spaces in this section. Throughout, let V be a C_σ -space. We will identify V , as in the previous section, with the C_σ -space $C_\sigma(K)$, where $K = \text{ext } V_1^*$ is the set of nonzero triple homomorphisms from V to \mathbb{C} .

Lemma 3.1. *Let V be separably injective. Given $a \in V$ of unit norm and the identity $e \in V(a)^{**}$, let $K(e) = \{\rho \in K : \rho(e) = 1\}$. Then there exists an element $v_a \in V$ such that $K(e) \subset K_a$ where*

$$K_a = \{\rho \in K : \rho(v_a) = 1\}$$

and K_a is weak* compact in V^* .

Proof. Since K_a is weak* closed in $K \cup \{0\}$, it is weak* compact. Let

$$\phi : C_0(S_a) \rightarrow V(a) \subset V$$

be the embedding in (2.2), where the triple spectrum $S_a \subset (0, 1]$ can be identified, via the evaluation map as usual, with the pure normal state space of $C_0(S_a)^{**}$.

Let χ_a be the constant function on S_a with value 1 and consider the separable subspace $C_0(S_a) + \mathbb{C}\chi_a$ of $\ell^\infty(S_a)$. By separable injectivity of V , the embedding $\phi : C_0(S_a) \rightarrow V(a) \subset V$ admits a norm preserving extension $\Phi : C_0(S_a) + \mathbb{C}\chi_a \rightarrow V$. Let

$$v_a = \Phi(\chi_a) \in V.$$

To complete the proof, we show that $\rho(v_a) = 1$ for each $\rho \in K(e)$.

We first observe that the sequence (r_n) of odd roots of the identity function ι_a in $C_0(S_a)$ converges pointwise to the function χ_a . Let $u_n = \phi(r_n) \in V(a)$.

Let $\rho \in K(e)$. Then the map $\rho \circ \phi : C_0(S_a) \rightarrow \mathbb{C}$ is a pure normal state of $C_0(S_a)^{**}$. Hence we have $\rho(u_n) = \rho \circ \phi(r_n) \in [0, 1]$ and $\lim_n \rho(u_n) = 1$.

The norm preserving extension Φ satisfies $\|\Phi(\chi_a)\| \leq 1$ and

$$\|\Phi(\chi_a) - 2u_n\| = \|\Phi(\chi_a) - 2\phi(r_n)\| = \|\Phi(\chi_a) - 2\Phi(r_n)\| \leq \|\chi_a - 2r_n\| \leq 1.$$

It follows that $|\rho(\Phi(\chi_a)) - 2\rho(u_n)| \leq 1$ for all n , which implies

$$|\rho(\Phi(\chi_a)) - 2| \leq 1.$$

Now $|\rho(\Phi(\chi_a))| \leq 1$ gives $\rho(v_a) = \rho(\Phi(\chi_a)) = 1$. \square

Our next task is to show that a separably injective C_σ -space V is actually linearly isometric to an abelian C^* -algebra. We adopt the following strategy. Since V is abelian, it has been noted in Section 1 that there is a composition series $(J_\lambda)_{0 \leq \lambda \leq \mu}$ of closed triple ideals in V such that for each ordinal $\lambda < \mu$, the quotient $J_{\lambda+1}/J_\lambda$ is linearly isometric to the C^* -algebra $C_0(X_\lambda)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space X_λ , and V^{**} is linearly isometric to the ℓ^∞ -sum $\bigoplus_{\lambda < \mu}^{\ell^\infty} (J_{\lambda+1}/J_\lambda)^{**} = \bigoplus_{\lambda < \mu}^{\ell^\infty} C_0(X_\lambda)^{**}$. By the uniqueness of predual, V^* is linearly isometric to the ℓ^1 -sum

$$\bigoplus_{\lambda < \mu}^{\ell^1} (J_{\lambda+1}/J_\lambda)^* = \bigoplus_{\lambda < \mu}^{\ell^1} C_0(X_\lambda)^*.$$

Given that V is separably injective, we will refine this construction to show that V is isometric to the abelian C^* -algebra $\bigoplus_{\lambda < \mu}^{c_0} C_0(X_\lambda)$.

Let V be separably injective and let $a \in V$ be of unit norm. Consider the closed subtriple $V(a)$ generated by a in V as well as the closed triple ideal J_a , the latter is linearly isometric to the C^* -algebra $C_0(X_a)$ as shown before, where $X_a = \{\rho \in K : \rho(e_a) = 1\}$ is weak* locally compact by Lemma 2.2 and Remark 2.3, and e_a is the identity of the von Neumann algebra J_a^{**} . By separable injectivity and Lemma 3.1, there exists $v_a \in V$ such that $\rho(v_a) = 1$ for each $\rho \in X_a$.

If $J_a = V$, then we are done. Otherwise we have the ℓ^∞ -sum

$$V^{**} = J_a^{**} \oplus_\infty (J_a^{**})^\square$$

where $(J_a^{**})^\square$ is a nonzero weak* closed triple ideal in V^{**} , orthogonal to J_a^{**} , that is, $J_a^{**} \square (J_a^{**})^\square = \{0\}$ (cf. [6, Lemma 3.3.16]). The quotient map $V^{**} \rightarrow V^{**}/J_a^{**}$ identifies $(J_a^{**})^\square$ with the quotient V^{**}/J_a^{**} and we can write

$$(3.1) \quad V^{**} = J_a^{**} \oplus_\infty (V^{**}/J_a^{**}) = C_0(X_a)^{**} \oplus_\infty (V/J_a)^{**}.$$

We have the ℓ^1 -sum $V^* = C_0(X_a)^* \oplus_1 (V/J_a)^*$ where J_a is an M-ideal in V . Consider the quotient map

$$x \in V \mapsto [x] := x + J_a \in [V] := V/J_a$$

which maps the closed unit ball of V onto the closed unit ball of V/J_a (cf. [1, Corollary 5.6]).

Pick $[b] = b + J_a \in [V]$ with unit norm. Let $V([b])$ and $J_{[b]}$ be respectively the closed subtriple and triple ideal generated by $[b]$ in $[V]$. We can repeat the previous arguments in the setting $V([b]) \subset J_{[b]} \subset [V]$ to deduce that $J_{[b]}$ is linearly isometric to some abelian C^* -algebra $C_0(X_{[b]})$ with

$$X_{[b]} = \{\theta \in \text{ext}[V]_1^* : \theta(e_{[b]}) = 1\}$$

where $X_{[b]}$ is locally compact in the weak* topology of $[V]^*$ by Lemma 2.2 and $e_{[b]}$ is the identity of $J_{[b]}^{**} \subset [V]^{**} = V^{**}/J_a^{**}$, which identifies with a tripotent $\tilde{e}_b \in (J_a^{**})^\square$ with $e_{[b]} = \tilde{e}_b + J_a^{**}$. Moreover, the quotient JB*-triple $[V] = V/J_a$ is abelian and separably injective by [2, Proposition 4.6], and hence Lemma 3.1 implies that there exists $v_{[b]} = \tilde{v}_b + J_a \in [V] = V/J_a$ such that $\theta(v_{[b]}) = 1$ for each $\theta \in X_{[b]}$.

The set $\text{ext}[V]_1^*$ consists of nonzero complex-valued triple homomorphisms on $[V]$, which can be lifted to nonzero complex triple homomorphisms on V via the

quotient map and we have

$$\text{ext } [V]_1^* = \{\bar{\rho} : \rho \in K, \rho(J_a) = \{0\}\}$$

where $\bar{\rho}(x + J_a) := \rho(x)$. Hence we have

$$X_{[b]} = \{\bar{\rho} : \rho \in K, \rho(a) = 0, \rho(\tilde{e}_b) = 1\}$$

and $\bar{\rho} \in X_{[b]}$ implies $\rho(b) = \bar{\rho}([b]) \neq 0$ and $\rho(\tilde{v}_b) = 1$.

A weak* convergent net in $[V]^*$ lifts to a weak* convergent net in V^* via the quotient map. Considering $X_{[b]} \subset C_0(X_{[b]})^* = J_{[b]}^*$ and Lemma 2.2, we see that a net $(\bar{\rho}_\gamma)$ in $X_{[b]}$ converges to $\bar{\rho} \in X_{[b]}$ in the weak* topology of $C_0(X_{[b]})^*$ if and only if (ρ_γ) weak* converges to ρ in K . The homeomorphism

$$\bar{\rho} \in X_{[b]} \mapsto \rho \in \{\rho \in K : \rho(a) = 0, \rho(\tilde{e}_b) = 1\}$$

enables us to identify these two spaces. We note that

$$(3.2) \quad \mathbb{T}X_a \cap \mathbb{T}X_{[b]} \subset \mathbb{T}X_a \cap \text{ext } [V]_1^* = \emptyset$$

where $\rho(a) \neq 0$ for all $\rho \in X_a$.

In the ℓ^1 -sum

$$(3.3) \quad V^* = J_a^* \oplus_1 (V/J_a)^* = C(X_a)^* \oplus_1 [V]^*,$$

each $\omega \in V^*$ admits a decomposition $\omega = \omega^1 + \omega^2$ in V^* with $\omega^2(J_a) = \{0\}$ and $\|\omega\| = \|\omega^1|_{J_a}\| + \|\omega^2\|$, which provides the identification of ω as an element $(\tilde{\omega}^1, \tilde{\omega}^2)$ in the ℓ^1 -sum, defined by

$$\tilde{\omega}^1 = \omega^1|_{J_a} \in J_a^* \quad \text{and} \quad \tilde{\omega}^2([\cdot]) = \omega^2(\cdot) \in (V/J_a)^*.$$

Hence for an extreme point $\omega \in \text{ext } V_1^* = K$, we have $\tilde{\omega}^1 = 0$ or $\tilde{\omega}^2 = 0$.

Given a net (ω_γ) in V^* weak* converging to a limit $\omega \in V^*$, it can be seen that the net $(\tilde{\omega}_\gamma^1)$ converges to $\tilde{\omega}^1$ in the $w(J_a^*, J_a)$ -topology and the net $(\tilde{\omega}_\gamma^2)$ converges to $\tilde{\omega}^2$ in the weak* topology of $(V/J_a)^*$. In particular, if the net (ω_γ) is in K and $\omega \in K$ with $\tilde{\omega}^j \neq 0$ ($j \in \{1, 2\}$), then the convergence of $(\tilde{\omega}_\gamma^j)$ to $\tilde{\omega}^j$ implies that $\tilde{\omega}_\gamma^j \neq 0$ eventually, and hence $\tilde{\omega}_\gamma^{j'} = 0$ for $j' \neq j$ eventually.

The closed unit ball of the ℓ^1 -sum in (3.3) has extreme points

$$(\mathbb{T}X_a, 0) \cup (0, \text{ext } [V]_1^*) := \{(\omega, 0) : \omega \in \mathbb{T}X_a\} \cup \{(0, \omega) : \omega \in \text{ext } [V]_1^*\}$$

and in the identification of (3.2), we have the disjoint union $K = \mathbb{T}X_a \cup \text{ext } [V]_1^*$. Given a net (ω_γ) in K weak* converging to some $\omega \in K$, and given either $\omega \in \mathbb{T}X_a$ or $\omega \in \text{ext } [V]_1^*$, the above observation implies that ω_γ belongs to the same set eventually.

Observe that $[J_b] = J_{[b]} = (J_b + J_a)/J_a$ and if $J_{[b]} \neq [V]$, we have the ℓ^∞ -sum

$$V^{**} = J_a^{**} \oplus_\infty ((J_b + J_a)/J_a)^{**} \oplus_\infty ([V]/J_{[b]})^{**} = C_0(X_a)^{**} \oplus_\infty C_0(X_{[b]})^{**} \oplus_\infty ([V]/J_{[b]})^{**}$$

where the quotient JB*-triple $[[V]] := [V]/J_{[b]}$ is separably injective, $\rho(v_a) = 1$ for $\rho \in X_a$ and $\rho(\tilde{v}_b) = 1$ for $\rho \in X_{[b]}$.

The closed unit ball of the ℓ^1 -sum

$$V^* = C_0(X_a)^* \oplus_1 C_0(X_{[b]})^* \oplus_1 ([V]/J_{[b]})^*$$

has extreme points

$$(\mathbb{T}X_a, 0, 0) \cup (0, \mathbb{T}X_{[b]}, 0) \cup (0, 0, \text{ext } [[V]]_1^*)$$

and we have the disjoint union $K = \text{ext } V_1^* = \mathbb{T}X_a \cup \mathbb{T}X_{[b]} \cup \text{ext } [[V]]_1^*$. Given a net (ω_γ) in K weak* converging to some $\omega \in K$, if ω belongs to one of the three

sets above, then repeating the arguments as before, ω_γ belongs to the same set eventually.

Now transfinite induction together with separable injectivity yields a composition series $(J_\lambda)_{0 \leq \lambda \leq \mu}$ of closed triple ideals in V , with $v_\lambda \in V$, such that V^* is linearly isometric to, and identifies with, the ℓ^1 -sum

$$\bigoplus_{\lambda < \mu}^{\ell^1} (J_{\lambda+1}/J_\lambda)^* = \bigoplus_{\lambda < \mu}^{\ell^1} C_0(X_\lambda)^*$$

where $\rho(v_\lambda) = 1$ for each $\rho \in X_\lambda$, and the pure normal state space X_λ of $(J_{\lambda+1}/J_\lambda)^{**}$ identifies with the set

$$\{\rho \in K : \rho(J_\lambda) = \{0\}, \rho(\tilde{e}_\lambda) = 1\}$$

in which \tilde{e}_λ is the identity of $(J_{\lambda+1}/J_\lambda)^{**}$. In this identification, we have the disjoint union

$$K = \bigcup_{\lambda < \mu} \mathbb{T}X_\lambda$$

and for a weak* convergent net (ω_γ) in K with limit $\omega \in \mathbb{T}X_\lambda$ for some λ , we have (ω_γ) in $\mathbb{T}X_\lambda$ eventually. As a consequence of Lemma 2.2, in the identification $X_\lambda \subset C_0(X_\lambda)^*$ and $X_\lambda \subset K$, the weak* convergence in $C_0(X_\lambda)^*$ of a net (ρ_γ) in X_λ to $\rho \in X_\lambda$ is the same as the weak* convergence in K .

Lemma 3.2. *Given that V is separably injective and in the above notation, the subset $X_\lambda \cup \{0\}$ of $K \cup \{0\}$ is weak* compact for all $\lambda < \mu$ and also, $\mathbb{T}X_\lambda$ is relatively weak* open in $K \cup \{0\}$.*

Proof. Let (ρ_γ) be a net in X_λ weak* converging to a nonzero limit $\omega \in V^*$. Then $\omega \in K$ and by the above remark, we must have $\omega \in \mathbb{T}X_\lambda$, say $\omega = \alpha\rho$ with $\alpha \in \mathbb{T}$ and $\rho \in X_\lambda$. Since X_λ is contained in the weak* compact set $\{\rho' \in K : \rho'(v_\lambda) = 1\}$, it follows that $\alpha = \alpha\rho(v_\lambda) = \lim_\gamma \rho_\gamma(v_\lambda) = 1$ and $\omega = \rho \in X_\lambda$. This proves that $X_\lambda \cup \{0\}$ is weak* closed in $K \cup \{0\}$ and hence weak* compact.

For the second assertion, let (ω_γ) be a net in $(K \cup \{0\}) \setminus \mathbb{T}X_\lambda$ weak* converging to some $\omega \in K$. Then again $\omega \notin \mathbb{T}X_\lambda$ for otherwise, the previous remark implies that ω_γ belongs to $\mathbb{T}X_\lambda$ eventually which is impossible. \square

The above construction enables us to show that a separably injective C_σ -space is isometric to an abelian C^* -algebra.

Theorem 3.3. *Let V be a separably injective C_σ -space. Then V is linearly isometric to an abelian C^* -algebra.*

Proof. Let $K = \text{ext } V_1^*$ and as shown previously, we have the ℓ^1 -sum

$$V^* = \bigoplus_{\lambda < \mu}^{\ell^1} C_0(X_\lambda)^*$$

with the disjoint union $K = \bigcup_{\lambda < \mu} \mathbb{T}X_\lambda$. For each $\lambda < \mu$, there is an element $v_\lambda \in V$ such that $\rho(v_\lambda) = 1$ for all $\rho \in X_\lambda$. We show that V is linearly isometric to the c_0 -sum $\bigoplus_{\lambda < \mu} C_0(X_\lambda)$ which would complete the proof.

We continue to identify V with $C_\sigma(K)$ in (2.1). By Lemma 3.2, each $f \in C_\sigma(K)$ restricts to a continuous function $f|_{X_\lambda} \in C_0(X_\lambda)$.

We show that the map

$$f \in V \approx C_\sigma(K) \mapsto (f|_{X_\lambda}) \in \bigoplus_{\lambda < \mu}^{c_0} C_0(X_\lambda)$$

is a surjective linear isometry.

To see that $(f|_{X_\lambda})$ indeed belongs to the c_0 -sum, we need to show $(\|f|_{X_\lambda}\|) \in c_0(\Lambda)$, where $\Lambda = [0, \mu)$. Let $\varepsilon > 0$. By Lemma 3.2, for each $\lambda \in \Lambda$, the set $\mathbb{T}X_\lambda$ is relatively weak* open in $K \cup \{0\}$. Since f vanishes at infinity on K , the set

$$U_\varepsilon = \{\omega \in K : |f(\omega)| < \varepsilon\} \cup \{0\}$$

is a relatively weak* open neighbourhood of 0 in the one-point compactification $K \cup \{0\}$ of K . We have

$$K \cup \{0\} = \bigcup_{\lambda \in \Lambda} \mathbb{T}X_\lambda \cup U_\varepsilon$$

and by weak* compactness, there are finitely many $\lambda_1, \dots, \lambda_n$ such that

$$K \cup \{0\} \subset \mathbb{T}X_{\lambda_1} \cup \dots \cup \mathbb{T}X_{\lambda_n} \cup U_\varepsilon.$$

It follows that $\|f|_{X_\lambda}\| \leq \varepsilon$ for $\lambda \notin \{\lambda_1, \dots, \lambda_n\}$ which proves $(\|f|_{X_\lambda}\|) \in c_0(\Lambda)$.

Since $K = \bigcup_{\lambda} \mathbb{T}X_\lambda$ and each $f \in C_\sigma(K)$ satisfies $f(\alpha\omega) = \alpha f(\omega)$ for $\alpha \in \mathbb{T}$ and $\omega \in K$, it is evident that the map is a linear isometry.

It remains to show that the map is surjective. Let $(g_\lambda) \in \bigoplus_{\lambda \in \Lambda}^{c_0} C_0(X_\lambda)$. Define a function $f : K \rightarrow \mathbb{C}$ by

$$f(\omega) = \alpha g_\lambda(\rho_\lambda) \quad \text{for } \omega = \alpha \rho_\lambda \in \mathbb{T}X_\lambda.$$

The function f is well-defined since the sets $\{\mathbb{T}X_\lambda\}_\lambda$ are mutually disjoint and each $\omega \in K$ has a unique representation $\omega = \alpha \rho \in \mathbb{T}X_\lambda$ for if $\alpha \rho = \beta \sigma \in \mathbb{T}X_\lambda$, we have $\alpha = \alpha \rho(v_\lambda) = \beta \sigma(v_\lambda) = \beta$, where X_λ is contained in the weak* compact set $\{\rho' \in K : \rho'(v_\lambda) = 1\}$.

We complete the proof by showing $f \in C_\sigma(K)$. We have readily $f(\alpha\omega) = \alpha f(\omega)$ for $\alpha \in \mathbb{T}$ and $\omega \in K$.

For continuity, let (ω_γ) be a net weak* converging to $\omega \in K$ and say, $\omega = \alpha \rho \in \mathbb{T}X_\lambda$ for some λ . By a previous remark, the net (ω_γ) is in $\mathbb{T}X_\lambda$ eventually. Therefore we have $\omega_\gamma = \alpha_\gamma \rho_\gamma$ with $\alpha_\gamma \in \mathbb{T}$ and $\rho_\gamma \in X_\lambda$ eventually. It follows that eventually $\alpha_\gamma = \alpha_\gamma \rho_\gamma(v_\lambda) \rightarrow \alpha \rho(v_\lambda) = \alpha$ and $\rho_\gamma \rightarrow \rho$. Hence we have

$$\lim_{\gamma} f(\omega_\gamma) = \lim_{\gamma} \alpha_\gamma g_\lambda(\rho_\gamma) = \alpha g_\lambda(\rho) = f(\omega).$$

Finally, for any $\varepsilon > 0$, there are finitely many $\lambda_1, \dots, \lambda_n$ such that $\|g_\lambda\| < \varepsilon$ for $\lambda \notin \{\lambda_1, \dots, \lambda_n\}$. For each λ_j with $j = 1, \dots, n$, there is a weak* compact set $E_j \subset X_{\lambda_j}$ such that $\{\rho \in X_{\lambda_j} : |g_{\lambda_j}(\rho)| \geq \varepsilon\} \subset E_j$. This gives

$$\{\omega \in K : |f(\omega)| \geq \varepsilon\} = \bigcup_{j=1}^n \mathbb{T}\{\rho \in X_{\lambda_j} : |g_{\lambda_j}(\rho)| \geq \varepsilon\} \subset \bigcup_{j=1}^n \mathbb{T}E_j \subset K$$

where the finite union $\bigcup_j \mathbb{T}E_j$ is weak* compact and therefore $f \in C_0(K)$. \square

Finally, by the characterisation of separably injective abelian C^* -algebras in [7, Theorem 3.5], together with Theorem 3.3, we conclude with the following main result of the paper.

Theorem 3.4. *Let V be a C_σ -space. The following conditions are equivalent.*

- (i) *V is separably injective.*
- (ii) *V is linearly isometric to the Banach space $C_0(S)$ of complex continuous functions vanishing at infinity on a substonean locally compact Hausdorff space S .*

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